AP Calculus BC
1999 Sample Student Responses

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B

CALCULUS BC

SECTION II

Time—1 hour and 30 minutes

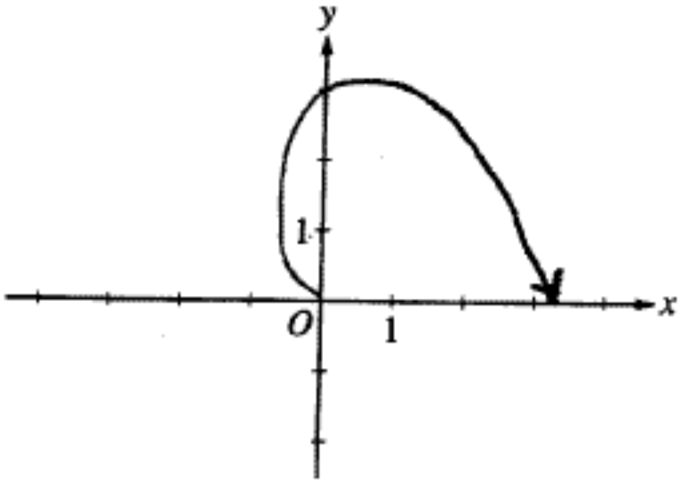
Number of problems—6

Percent of total grade—50

REMEMBER TO SHOW YOUR SETUPS AS DESCRIBED IN THE GENERAL INSTRUCTIONS.

1. A particle moves in the xy-plane so that its position at any time t, $0 \le t \le \pi$, is given by $x(t) = \frac{t^2}{2} - \ln(1 + t)$ and $y(t) = 3 \sin t$.

   (a) Sketch the path of the particle in the xy-plane below. Indicate the direction of motion along the path.



Continue problem 1 on page 5.

B

(b) At what time t, $0 \le t \le \pi$, does $x(t)$ attain its minimum value? What is the position $(x(t),\ y(t))$ of the particle at this time?

$$x(t) = \frac{t^2}{2} - \ln(1+t)$$

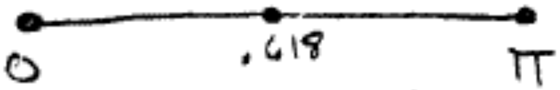
$$x'(t) = t - \frac{1}{1+t}$$

$$0 = t - \frac{1}{1+t}$$

$$t(t+1) - 1 = 0$$

$$t^2 + t - 1 = 0$$

$$\boxed{t = .618}$$



$x'(t)$ − +

$x(t)$ dec. inc

$\Rightarrow \therefore$ Minimum at

$t = .618$

$$P(.618) = \left(\frac{(.618)^2}{2} - \ln(1+.618),\ 3\sin(.6\right.$$

$$P(.618) = (-.290,\ 1.738)$$

(c) At what time t, $0 < t < \pi$, is the particle on the y-axis? Find the speed and the acceleration vector of the particle at this time.

When $x=0$, the particle is on the y axis

$$\Rightarrow x(t) = 0 \Rightarrow \frac{t^2}{2} - \ln(1+t) = 0$$

$\therefore$ when $t = 1.286$ the particle is on the y-axis

$$\text{Velocity}_x = x'(t) = t - \frac{1}{1+t} \qquad \text{Velocity}_y = 3\cos t$$

$$\text{Speed} = \sqrt{x'(t)^2 + y'(t)^2}$$

$$= \sqrt{\left(t - \frac{1}{1+t}\right)^2 + (3\cos t)^2}$$

$$\text{Speed at } (t = 1.286) = 1.196$$

$$\text{Acceleration} = (\text{Velocity})'$$

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$$A(t) = \left(1 + \frac{1}{(1+t)^2},\ -3\sin t\right)$$

$$A(1.286) = (1.191,\ -2.879)$$

D

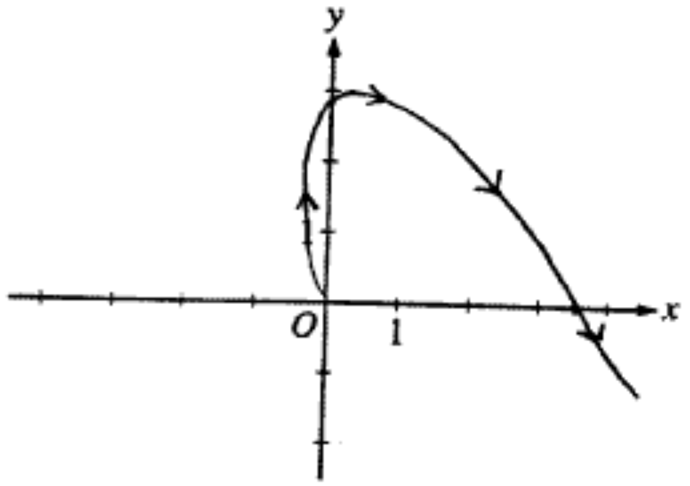
CALCULUS BC
SECTION II

Time—1 hour and 30 minutes
Number of problems—6
Percent of total grade—50

REMEMBER TO SHOW YOUR SETUPS AS DESCRIBED IN THE GENERAL INSTRUCTIONS.

1. A particle moves in the xy-plane so that its position at any time t, $0 \le t \le \pi$, is given by $x(t) = \frac{t^2}{2} - \ln(1 + t)$ and $y(t) = 3 \sin t$.

   (a) Sketch the path of the particle in the xy-plane below. Indicate the direction of motion along the path.



Continue problem 1 on page 5.

D

(b) At what time t, $0 \le t \le \pi$, does $x(t)$ attain its minimum value? What is the position $(x(t), y(t))$ of the particle at this time?

$$x(t)=\frac{t^2}{2}-\ln(1+t)$$

$$\frac{dx}{dt}=t-\frac{1}{1+t}$$

$$0=t-\frac{1}{1+t}$$

$$t=\frac{1}{1+t}$$

$$t^2+t=1$$

$$t^2+t-1=0$$

$$t=\frac{-1+\sqrt{5}}{2}$$ by quadratic formula

$x(t)=-.290$

$y(t)=1.73$

position is $(-.290, 1.73)$

(c) At what time t, $0 < t < \pi$, is the particle on the y-axis? Find the speed and the acceleration vector of the particle at this time.

$$x(t)=\frac{t^2}{2}-\ln(1+t)$$

$$0=\frac{t^2}{2}-\ln(1+t)$$

$$\ln(1+t)=\frac{t^2}{2}$$

$t=0$ at 0 and 1.285

t is on the y axes at 1.285 for $0<t<\pi$

$$v(t)=\left(t-\frac{1}{1+t}, 3\cos t\right)$$

$$a(t)=\left(1+\frac{1}{(1+t)^2}, -3\sin t\right)$$

$$a(t)=(1.191, -2.879)$$

$$\frac{dx}{dt}=t-\frac{1}{1+t} \qquad \frac{dy}{dt}=3\cos t$$

$$v(t)=\left(t-\frac{1}{1+t}, 3\cos t\right)$$

speed = magnitude of velocity

$\frac{dx}{dt}$ .848 $\quad \frac{dy}{dt}=.843$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=1.196=\text{speed}$$

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F

CALCULUS BC

SECTION II

Time—1 hour and 30 minutes
Number of problems—6
Percent of total grade—50

REMEMBER TO SHOW YOUR SETUPS AS DESCRIBED IN THE GENERAL INSTRUCTIONS.

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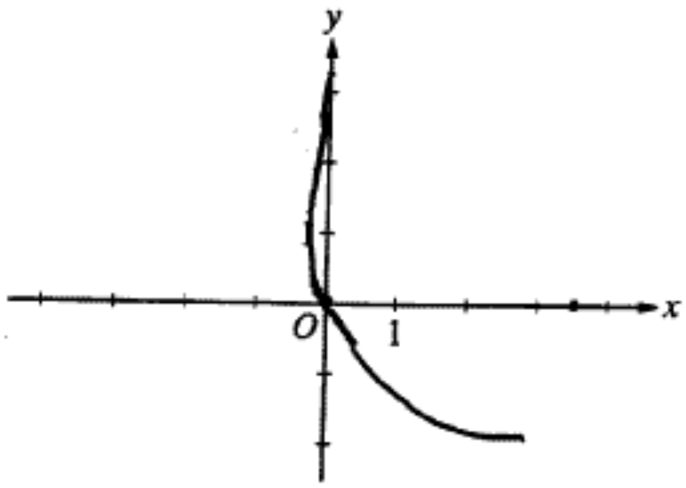
   (a) Sketch the path of the particle in the xy-plane below. Indicate the direction of motion along the path.

X

$\frac{e}{e^{\frac{t^2}{2}}} = 1+t$



Continue problem 1 on page 5.

F

(b) At what time t, $0 \le t \le \pi$, does $x(t)$ attain its minimum value? What is the position $(x(t),\ y(t))$ of the particle at this time?

$$\frac{dx}{dt} = t - \frac{1}{1+t} = 0$$

$t = 0.618$

$(-0.29, 1.738)$

(c) At what time t, $0 < t < \pi$, is the particle on the y-axis? Find the speed and the acceleration vector of the particle at this time.

$t = 1.286$

$\breve{v} = (0.849,\ 0.843)$

$\breve{a} = (1.191, -2.879)$

$1 + \frac{1}{(1+t)^2}$

$-3\sin t$

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AP Calculus BC
1999 Sample Student Responses

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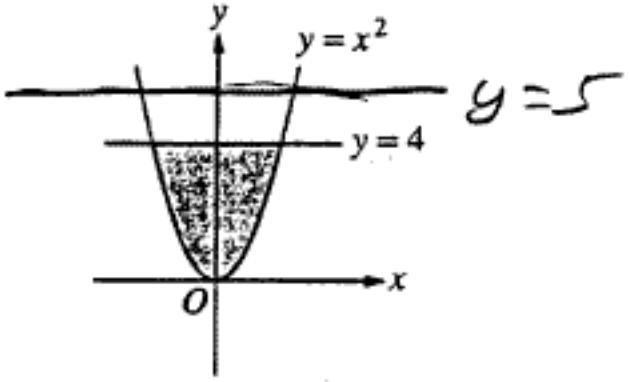

2. The shaded region, R, is bounded by the graph of $y = x^2$ and the line $y = 4$, as shown in the figure above.

(a) Find the area of R.

$$\int_{-2}^{2} 4 - x^2\,dx = \frac{32}{3}$$

(b) Find the volume of the solid generated by revolving R about the x-axis.

$$\pi\int_{-2}^{2} 4^2 - x^4\,dx = \pi\left\{\left[16x - \frac{1}{5}x^5\right]_{-2}^{2}\right\} =$$

$$51{,}2\pi$$

Continue problem 2 on page 7.

A_2

(c) There exists a number k, $k > 4$, such that when R is revolved about the line $y = k$, the resulting solid has the same volume as the solid in part (b). Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.

$$\pi\int_{-2}^{2} (K-x^2)^2-(K-4)^2 dx = 51,2\pi$$

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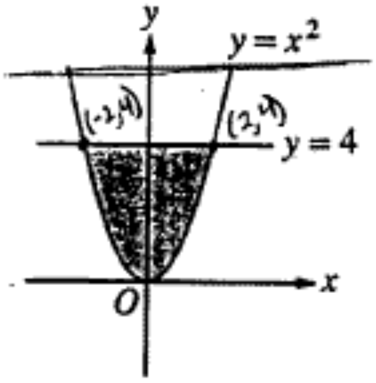


2. The shaded region, R, is bounded by the graph of $y = x^2$ and the line $y = 4$, as shown in the figure above.

(a) Find the area of R.

$$A = \int_{-2}^{2}(4-x^2)\,dx = 2\int_{0}^{2}(4-x^2)\,dx = 2\left[4x-\frac{1}{3}x^3\right]_0^2$$

$$A = 2\left(4(2)-\frac{1}{3}(2)^3 \cancel{-0-0}\right) = 2\left(8-\frac{8}{3}\right) = \frac{32}{3}$$

$$\boxed{A_R = \frac{32}{3}}$$

(b) Find the volume of the solid generated by revolving R about the x-axis.

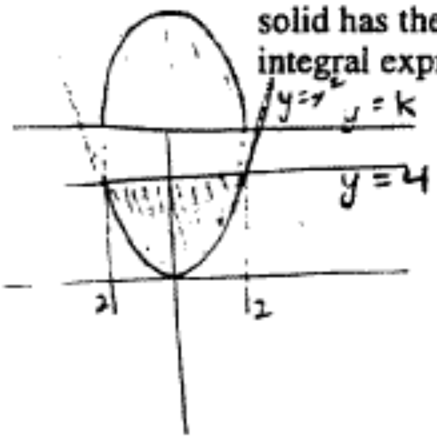
$$V = \pi\int_{-2}^{2}\left(4^2-(x^2)^2\right)dx = 2\pi\int_0^2(16-x^4)\,dx = 2\pi\left[16x-\frac{1}{5}x^5\right]_0^2$$

$$V = 2\pi\left[16(2)-\frac{1}{5}(2)^5 \cancel{-0-0}\right] = 2\pi\left(32-\frac{32}{5}\right) = \frac{256\pi}{5}$$

$$\boxed{V = \frac{256\pi}{5} = 160.850}$$

Continue problem 2 on page 7.

(c) There exists a number k, $k > 4$, such that when R is revolved about the line $y = k$, the resulting solid has the same volume as the solid in part (b). Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.



$$V = \pi\int_{-2}^{2}\left(\left(k-4\right)^2 - \left(k-x^2\right)^2\right)dx$$

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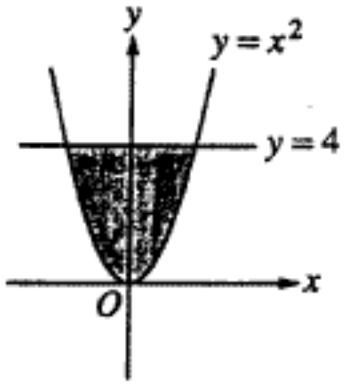
E₁



2. The shaded region, R, is bounded by the graph of $y = x^2$ and the line $y = 4$, as shown in the figure above.

(a) Find the area of R.

$$\int_{-2}^{2} x^2 - 4 = \Big[_{-2}^{0} \frac{x^3}{3} - 4x + \Big[_{0}^{2} \frac{x^3}{3} - 4x = \boxed{\frac{32}{3}}$$

(b) Find the volume of the solid generated by revolving R about the x-axis.

$$V = \pi \int_{-2}^{2} (x^2 - 4)^2 dx \quad \pi \int x^4 - 16 \, dx$$

$$V = \pi \Big[_{-2}^{2} \frac{x^5}{5} - 16x \Rightarrow \boxed{V \approx 59.733\pi}$$

Continue problem 2 on page 7.

(c) There exists a number k, $k > 4$, such that when R is revolved about the line $y = k$, the resulting solid has the same volume as the solid in part (b). Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.

$$\boxed{59.733\pi = \pi \int_{-2}^{2} (x^2 - k)^2 \, dx}$$

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B

t (hours)	$R(t)$ (gallons per hour)
0	9.6
3	10.4
→ 6	10.8
9	11.2
→ 12	11.4
15	11.3
→ 18	10.7
21	10.2
→ 24	9.6

3. The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t. The table above shows the rate as measured every 3 hours for a 24-hour period.

(a) Use a midpoint Riemann sum with 4 subdivisions of equal length to approximate $\int_0^{24} R(t)dt$. Using correct units, explain the meaning of your answer in terms of water flow.

$\sum_{i=1}^{4} R(c_i)6 \quad 10.4 \cdot 6 + 11.2 \cdot 6 + 11.3 \cdot 6 + 10.2 \cdot 6$

where c_i = midpoint of interval

$\boxed{= 258.6 \text{ gallons}}$

$(t = 3, 9, 15, 21)$

= # of gallons of water to flow out of a pipe from $t = 0$ to $t = 24$

(b) Is there some time t, $0 < t < 24$, such that $R'(t) = 0$? Justify your answer.

Yes → $\frac{R(24) - R(0)}{24 - 0} = 0$, therefore, by the Mean Value Theorem, there is some t in (0, 24) such that $R'(t) = 0$

-8-

Continue problem 3 on page 9.

B

(c) The rate of water flow $R(t)$ can be approximated by $Q(t) = \frac{1}{79}\left(768 + 23t - t^2\right)$.
Use $Q(t)$ to approximate the average rate of water flow during the 24-hour time period.

Indicate units of measure.

$$\frac{\frac{1}{79}\int_0^{24}(768+23t-t^2)\,dt}{24}$$

$\approx$ 10.7848 gallons / hour

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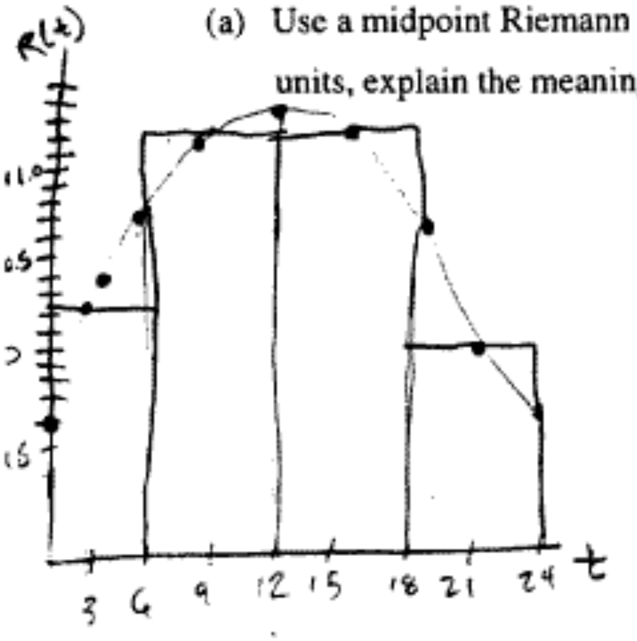
D 7

t (hours)	$R(t)$ (gallons per hour)
0	9.6
3	10.4
6	10.8
9	11.2
12	11.4
15	11.3
18	10.7
21	10.2
24	9.6

3. The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t. The table above shows the rate as measured every 3 hours for a 24-hour period.

(a) Use a midpoint Riemann sum with 4 subdivisions of equal length to approximate $\int_0^{24} R(t)dt$. Using correct units, explain the meaning of your answer in terms of water flow.



b·h

$(2\cdot 10.4)+(2\cdot 11.2)+(2\cdot 11.3)+(2\cdot 10.2)$

$\int_0^{24} R(t)\,dt \approx 86.2$ gallons

It means that 86.2 gallons of water flowed out of the pipe for that 24 hour period.

(b) Is there some time t, $0 < t < 24$, such that $R'(t) = 0$? Justify your answer.

Yes there is. At approx. $t \approx 12$ the slope of a tangent line to that point is 0. $\therefore$ $R'(t) = 0$ at that point.

Continue problem 3 on page 9.

D 7

(c) The rate of water flow $R(t)$ can be approximated by $Q(t) = \frac{1}{79}\left(768 + 23t - t^2\right)$. Use $Q(t)$ to approximate the average rate of water flow during the 24-hour time period. Indicate units of measure.

$$\text{Total Flow} = \int_0^{24} Q(t)\,dt$$

$$= 258.83544 \text{ gallons}$$

$$\text{average rate} = \frac{258.83544 \text{ gallons}}{24 \text{ hours}}$$

$$= 10.785 \text{ gallons/hour}$$

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E

t (hours)	$R(t)$ (gallons per hour)
0	9.6
3	10.4
6	10.8
9	11.2
12	11.4
15	11.3
18	10.7
21	10.2
24	9.6

3. The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t. The table above shows the rate as measured every 3 hours for a 24-hour period.

(a) Use a midpoint Riemann sum with 4 subdivisions of equal length to approximate $\int_0^{24} R(t)\,dt$. Using correct units, explain the meaning of your answer in terms of water flow.

$$RS = \frac{b-a}{n}\left[f(x_1)+f(x_2)+f(x_3)+\cdots f(x_n)\right]$$

$$RS = \frac{24}{4}\left[f(3)+f(9)+f(15)+f(21)\right]$$

$$RS = 6\ [10.4 + 11.2 + 11.3 + 10.2]$$

RS = 258.600 gallons

after 24 hours 258.600 gallons of water have flowed from the pipe

(b) Is there some time t, $0 < t < 24$, such that $R'(t) = 0$? Justify your answer.

$R'(t)$ is the slope of $R(t)$ if $R(t)$ is a velocity then $R'(t)$ is the acceleration or change in velocity between time $t=12$ and time $t=15$ the change in velocity changes from positive to negative so $R'(t)$ must $=0$ at some time t $12 \le t \le 15$ which is between 0 and 24

Continue problem 3 on page 9.

E

(c) The rate of water flow $R(t)$ can be approximated by $Q(t) = \frac{1}{79}\left(768 + 23t - t^2\right)$.
Use $Q(t)$ to approximate the average rate of water flow during the 24-hour time period.

Indicate units of measure.

$$\text{Avg Rate} \doteq \frac{Q(24) - Q(0)}{24 - 0}$$

$$Q(24) = \frac{1}{79}\left(768 + 23(24) - (24)^2\right)$$

$$Q(24) = 9.418$$

$$Q(0) = \frac{1}{79}\left(768 + 23(0) - (0)^2\right)$$

$$Q(0) = 9.722$$

$$\text{Avg Rate} \doteq \frac{9.418 - 9.722}{24}$$

Avg Rate $\doteq$ $-.013$ gallons per hour

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4. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

(a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

$$T_3(x) = -3 + 5(x-2) + \frac{3}{2}(x-2)^2 + -\frac{4}{3}(x-2)^3$$

$$f(1.5) \approx -3 + 5(1.5-2) + \frac{3}{2}(1.5-2)^2 - \frac{4}{3}(1.5-2)^3$$

$$= -4.958$$

(b) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \leq 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \neq -5$.

$R_3(1.5) = \frac{f^{(4)}(z)}{4!}(1.5-2)^4$ for some z, $1.5 \leq z \leq 2$.

Thus, $R_3(1.5) \leq \frac{3}{4!}(1.5-2)^4 = ,0078125$

Thus, $-4.958 - .0078125 \leq f(1.5) \leq -4.958 + .0078125$

$$-4.966 \leq f(1.5) \leq -4.950$$

Thus, $f(1.5) \neq -5$

Continue problem 4 on page 11.

(c) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f\left(x^2 + 2\right)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

$$P(x) = -3 + 5x^2 + \frac{3}{2}x^4$$

Since the coefficent of x is 0, $\frac{g'(0)}{1!} = 0$, so $g'(0) = 0$.

Since the coefficent of x^2 is 5, $\frac{g''(0)}{2!} = 5$, so $g''(0) = 10$

Thus, since $g''(0)$ is positive and $g'(0) = 0$, $P(x)$ must have a relative minimum at $x = 0$ by the second derivative test.

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D_1

4. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

(a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

$$f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$

$$= -3 + 5(x-2) + \frac{3}{2!}(x-2)^2 - \frac{8}{3!}(x-2)^3$$

$$f(1.5) = -4.958$$

(b) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \ne -5$.

$$|error| \le a_{n+1}$$

$$|error| \le a_4$$

$$|error| \le \frac{3}{4!}(x-2)^4$$

$$|error| \le \frac{3}{24}(1.5-2)^4$$

$$|error| \le .0078$$

The truncation error is no greater than .0078

$$\therefore \quad -4.958 - .0078 < f(1.5) \le -4.958 + .0078$$

$$-4.9658 \le f(1.5) \le -4.9502$$

$$\therefore \quad f(1.5) \ne -5$$

Continue problem 4 on page 11.

(c) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f\left(x^2 + 2\right)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

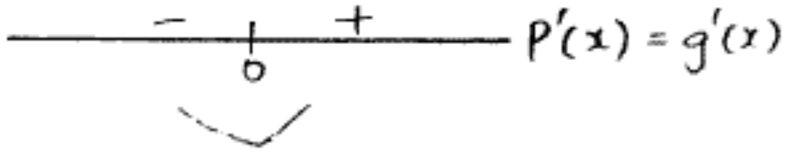
$$P(x) = -3 + 5(x^2+2-2) + \frac{3}{2!}(x^2+2-2)^2$$

$$= -3 + 5x^2 + \frac{3}{2!}x^4$$

$$P'(x) = 10x + 6x^3$$

$$0 = 2x(5 + 3x^2)$$

$$x = 0$$



∴ $g(x)$ has a relative minimum at $x = 0$

4. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

(a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

$$\boxed{f(x) = -3 + 5(x-2) + \frac{3(x-2)^2}{2!} - \frac{8(x-2)^3}{3!}}$$

$$f(1.5) = -3 + 5(1.5-2) + \frac{3(1.5-2)^2}{2!} - \frac{8(x-2)^3}{3!}$$

$$\boxed{f(1.5) = -4.958}$$

(b) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \ne -5$.

$$f^{3}(x) > f^{4}(x)$$

$$\frac{-8(-.5)^3}{3!} > f\frac{^{4}(-.5)^4}{4!}$$

$$1.667 > .002604\,(f^4)$$

since there is an error of at least .002604, $f(1.5) \ne -5$.

Continue problem 4 on page 11.

(c) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f\left(x^2 + 2\right)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

$g(x) = f(x^2+2)$

$P(x) = -3 + 5(x^2+2-2) + \frac{3(x^2+2-2)^2}{2!}$

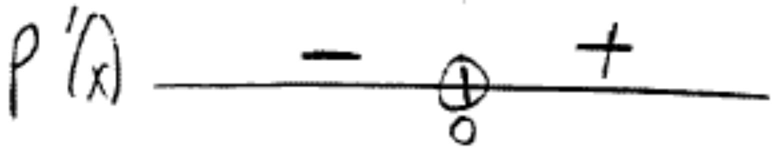
$\boxed{P(x) = -3 + 5x^2 + \frac{3x^4}{2!}}$

$P'(x) = 10x + \frac{12x^3}{2}$

$P''(x) = 10x + 6x^3$

$0 = 2x(5 + 3x^2)$

$x = 0$ other 2 roots are complex



Because $P'(x)$ has a relative min at $x=0$, so does $g(x)$

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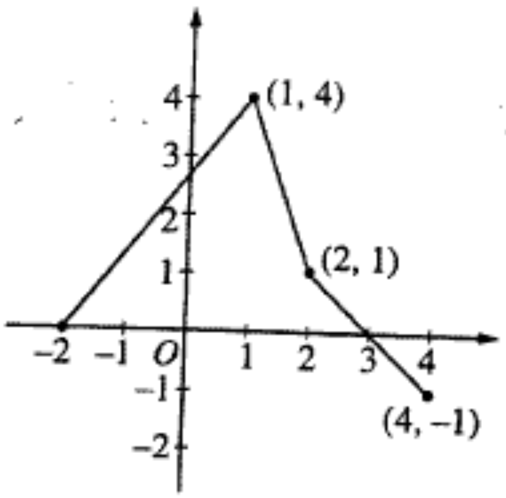
B



5. The graph of the function f, consisting of three line segments, is given above. Let $g(x) = \int_1^x f(t)dt$.

(a) Compute $g(4)$ and $g(-2)$.

$$g(4) = \int_1^4 f(t)dt = \int_1^2 f(t)\,dt + \int_2^4 f(t)\,dt$$

$$= \frac{5}{2} + 0 = \frac{5}{2}$$

$$g(-2) = \int_1^{-2} f(t)dt = -\int_{-2}^1 f(t)\,dt = -(\tfrac{1}{2}(3)(4)) = -6$$

(b) Find the instantaneous rate of change of g, with respect to x, at $x = 1$.

$$g'(x) = f(x)$$

$$g'(1) = f(1) = 4$$

Continue problem 5 on page 13.

(c) Find the absolute minimum value of g on the closed interval $[-2, 4]$. Justify your answer.

absolute minimum could occur at endpoint or when $g'(x)=0$

	x	$g(x)$
endpt / $g'(x)=0$	-2	-6
$g'(x)=0$	3	3
endpt	4	$\frac{5}{2}$

since $g(-2) < g(3)$ and $g(-2) < g(4)$ the absolute minimum occurs at -2 and is $g(-2) = -6$.

(d) The second derivative of g is not defined at $x = 1$ and $x = 2$. How many of these values are x-coordinates of points of inflection of the graph of g? Justify your answer.

$x=1$ is an inflection point because $g''(x) > 0$ for $x < 1$ and $g''(x) < 0$ for $x > 1$. $x=2$ is not an inflection point because $g''(x) < 0$ for $x < 2$ and $x > 2$.

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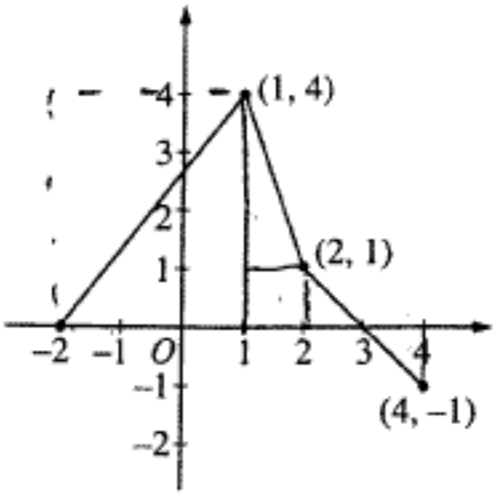
D



5. The graph of the function f, consisting of three line segments, is given above. Let $g(x) = \int_1^x f(t)dt$.

(a) Compute $g(4)$ and $g(-2)$.

$g(x) = \int_1^x f(t)\,dt$

$g(4) = \int_1^4 f(t)\,dt = 1 + \frac{3}{2} + \frac{1}{2} - \frac{1}{2} \Rightarrow$

$\boxed{g(4) = \frac{5}{2}}$

$g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^1 f(t)\,dt = -\left[\frac{3 \times 4}{2}\right] = -6$

$\boxed{g(-2) = -6}$

(b) Find the instantaneous rate of change of g, with respect to x, at $x = 1$.

$\frac{dg}{dt} = \frac{d}{dt}\int_1^x f(t)\,dt$

$\frac{dg}{dt} = f(t)\Big|_x$

$\boxed{\frac{d}{dt} g(1) = 4}$

Continue problem 5 on page 13.

(c) Find the absolute minimum value of g on the closed interval $[-2, 4]$. Justify your answer.

$g' = f(t) = 0$

$t = 3$

f $\quad \begin{array}{c} 3 \\ \hline + + + + 0 - - - \\ \text{pos} \mid \text{neg} \end{array}$

$t > 3$, g increasing

$t < 3$, g decreasing

$\because t = 3$ is rel <u>max</u>

⇒ absolute min must be one of the endpoints

OR $\begin{array}{l} x = 4 \\ x = -2 \end{array}$

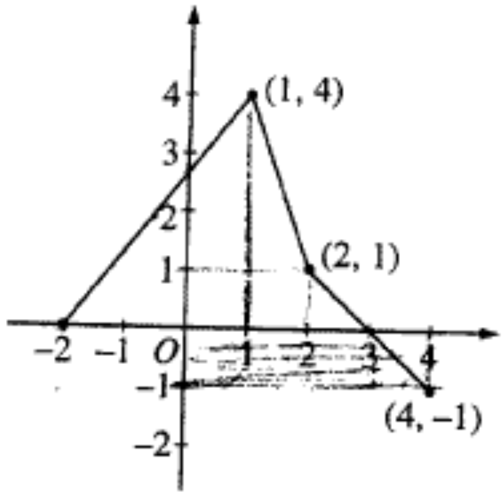
$$-\int_{-2}^{1} f(t)\,dt < \int_{1}^{4} f(t)\,dt$$

$\boxed{\text{x = -2 is absolute minimum because the area between f(t) and the x axis in the interval [-2,1] by -1 is clearly less than the area between f(t) and the x axis on the interval [1,4].}}$

(d) The second derivative of g is not defined at $x = 1$ and $x = 2$. How many of these values are x-coordinates of points of inflection of the graph of g ? Justify your answer.

x=1 is a point of inflection because the slope of f(t) [which equals $g'(x)$] changes from positive to negative at x=1. At x=2, the slope of f(t) stays positive for $1<x<2$ and $2<x<4$.

E



5. The graph of the function f, consisting of $\overset{-4}{\text{three}}$ line segments, is given above. Let $g(x) = \int_1^x f(t)dt$.

(a) Compute $g(4)$ and $g(-2)$.

$g(4) = \int_1^4 f(t)dt$

~~$A_{⏢} = \frac{1}{2}(b_1+b_2)$~~ h

~~$A_{⏢} = \frac{1}{2}($~~

~~$g(1) = \int_0^1 (\frac{4}{3}x$~~

4 $A_{⏢} = \frac{1}{2}(b_1+b_2)h$

$A_{⏢_1} = \frac{1}{2}(6.5) \times 1$

$A_{⏢_1} = 3.25$

$A_{⏢_2} = \frac{1}{2}(4+1)(1)$

$A_{⏢_2} = 2.5$

$g(4) = 3.25 + 2.5 + 1$

$\boxed{g(4) = 6.75}$

$g(-2) = \int_1^{-2} f(t)dt$

4 $A_{\Delta} = \frac{1}{2}bh$

$A_{\Delta} = \frac{1}{2}(2 \cdot 2.5)$

$A_{\Delta} = 5$

$\boxed{g(-2) = 5}$

(b) Find the instantaneous rate of change of g, with respect to x, at $x = 1$.

$g'(x) = f(x)$

$g'(1) = f(1)$

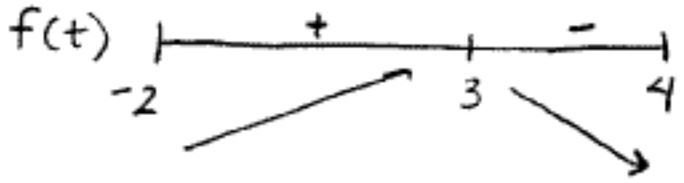
$f(1) = 4$

$\boxed{\text{Instantaneous rate of change} = 4}$

Continue problem 5 on page 13.

(c) Find the absolute minimum value of g on the closed interval $[-2, 4]$. Justify your answer.

$g'(x) = f(t)$



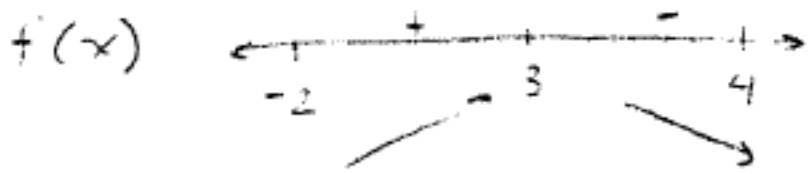
$f(-2) = 0$

$f(4) = -1$

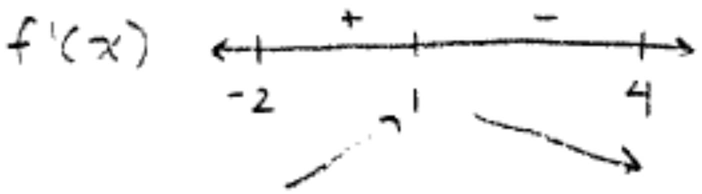
∴ There is an absolute minimum for $g(x)$ over $[-2,4]$ at $x = 4$

(d) The second derivative of g is not defined at $x = 1$ and $x = 2$. How many of these values are x-coordinates of points of inflection of the graph of g ? Justify your answer.

$g'(x) = f(x)$



$f'(x) = g''(x)$



$x = 1$ is a pt of inflection for $g(x)$

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A1

6. Let f be the function whose graph goes through the point (3, 6) and whose derivative is given by $f'(x) = \frac{1 + e^x}{x^2}$.

(a) Write an equation of the line tangent to the graph of f at $x = 3$ and use it to approximate $f(3.1)$.

slope @ $x=3$: $\frac{1+e^3}{9}$

$$\boxed{y-6=\left(\frac{1+e^3}{9}\right)(x-3)}$$

$$y-6=\left(\frac{1+e^3}{9}\right)(3.1-3)$$

$$f(3.1) \approx 6+\left(\frac{1+e^3}{9}\right)(.1)$$

$$\boxed{f(3.1) \approx 6.234}$$

(b) Use Euler's method, starting at $x = 3$ with a step size of 0.05, to approximate $f(3.1)$. Use f'' to explain why this approximation is less than $f(3.1)$.

x	y	slope	Δy
3	6	$\frac{1+e^3}{9}$	.117
3.05	6.117	2.377	
3.1	6.236		

$$\boxed{f(3.1) \approx 6.236}$$

$$f''(x) = \frac{x^2(e^x)-(1+e^x)2x}{x^4}$$

$f''(x)$ ⊢ + →
3 ccu

since $f''(x)$ is positive when $x > 3$ the graph of f is concave up, thus the tangent lines are below the actual graph of f and the values found by using the tangent lines are lower than the actual values

Continue problem 6 on page 15.

(c) Use $\int_3^{3.1} f'(x)dx$ to evaluate $f(3.1)$.

$\int_3^{3.1} f'(x)$

$= f(3.1) - f(3)$ $f(3)=6$

$\int_3^{3.1} \frac{1+e^x}{x^2}$

$= .238$

$.238 = f(3.1) - 6$

$\boxed{f(3.1) = 6.238}$

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D_1

6. Let f be the function whose graph goes through the point (3, 6) and whose derivative is given by $f'(x) = \frac{1 + e^x}{x^2}$.

(a) Write an equation of the line tangent to the graph of f at $x = 3$ and use it to approximate $f(3.1)$.

$$f'(3) = \frac{1+e^3}{9} \approx 2.343$$

tangeant line $\rightarrow$ $\underbrace{y-6 = 2.343(x-3)}_{\text{sub in}}$ ‖

sub in

$x = 3.1$

$$y-6 = 2.343(3.1-3)$$

$$y \approx f(3.1) \approx 6.234 \text{ //}$$

(b) Use Euler's method, starting at $x = 3$ with a step size of 0.05, to approximate $f(3.1)$. Use f'' to explain why this approximation is less than $f(3.1)$.

at $(3,6)$ slope $= \frac{1+e^3}{9} = 2.343$

$\downarrow$

$(3.05, 6+0.05(2.343))$

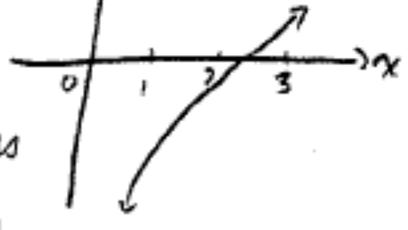
$= (3.05, 6.117)$ slope $= \frac{1+e^{3.05}}{3.05^2} = 2.377$

$\downarrow$

$(3.1, 6.117+0.05(2.377))$

$= (3.1, 6.236) \longrightarrow f(3.1) \approx 6.236$ //

$f''(x) = \frac{x^2e^x - 2x(1+e^x)}{x^4}$ $\rightarrow$ graph on calculator



The graph of $f''(x)$ is positive for $3 \leq x \leq 3.1$, which means f is concave up () ∴ Any tangeant line to f would lie under the graph, making the approximation less than the actual value //

Continue problem 6 on page 15.

D_2

(c) Use $\int_3^{3.1} f'(x)dx$ to evaluate $f(3.1)$.

$$\int_3^{3.1} \frac{1+e^x}{x^2}\,dx$$

evaluate in calculator ...

$$\approx 0.2378$$

$$f(3.1) \approx \frac{1}{3.1-3} \int_3^{3.1} f'(x)\,dx$$

$$\doteq \frac{1}{0.1}(0.2378)$$

$$= 2.378$$

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E₁

6. Let f be the function whose graph goes through the point (3, 6) and whose derivative is given by $f'(x) = \frac{1 + e^x}{x^2}$.

(a) Write an equation of the line tangent to the graph of f at $x = 3$ and use it to approximate $f(3.1)$.

$f'(x) = \frac{1+e^x}{x^2}$ $\quad f'(3) = \frac{1+e^3}{9} = m$ tangent

$\boxed{y-6 = \frac{1+e^3}{9}(x-3)}$ = tangent line

$y-6 = \frac{1+e^3}{9}(3.1-3)$

$y-6 = \frac{1+e^3}{9}(0.1)$

$\boxed{y \approx 6.234 \text{ at } x=3.1 \text{ by approximation}}$

(b) Use Euler's method, starting at $x = 3$ with a step size of 0.05, to approximate $f(3.1)$. Use f'' to explain why this approximation is less than $f(3.1)$.

$x_0 = 3 \quad y_0 = 6$

$x_1 = 3.05$ $\quad y_1 = 6 + 0.05\left(\frac{1+e^3}{3^2}\right)$

$y_1 = 6.1171418718$

$x_2 = 3.1$ $\quad y_2 = 6.1171418718 + 0.05\left(\frac{1+e^{3.05}}{(3.05)^2}\right)$

$\boxed{f(3.1) \approx y_2 = 6.236}$

This approximation is less than f(3.1) because we are underapproximating it.

Continue problem 6 on page 15.

(c) Use $\int_3^{3.1} f'(x)dx$ to evaluate $f(3.1)$.

$$\int_3^{3.1} \frac{1+e^x}{x^2}\, dy$$

$$\int_3^{3.1} \frac{1}{x^2}(1+e^x)\, dx$$

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